Assumpte: two corrections
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Give the free group F:= <x,y| > a bi-ordering. (For example, Magnus: Let t,u be non-commuting variables and let Z<<t,u>> be the power-series ring. The group of units of Z<<t,u>> with constant term 1 is bi-ordered with positive cone the set of terms with positive "first" coefficent. Here x:= 1+t, y:=1+u are independent, since x^{m_1}y^{n_1}...x^{m_s}y^{n_s} = (1+m_1t+...)(1+n_1u+...)...(1+m_st+...)(1+n_su+...) and if no m_i or n_i is zero the coefficient of (tu)^s is m_1n_1...m_sn_s which is nonzero.)

Let U (for universe) be the Cayley tree of F with edge set EU:= F \times {(1,x) ,(1,y)} given the lexicographic order with   (1,x) < (1,y).
Notice that for all g,h \in F and e,f \in EU, if g < h then ge < he, and if e < f then ge < gf.

Throughout, let G be a non-trivial f.g. subgroup of F.

We shall construct a canonical G-subset B(G) of EU (whose elements will be called G-bridges) which will have the following properties:
1. Each component of the G-forest U - B(G) will have G-stabilizer of rank exactly 1.
It then follows from Bass-Serre Theory that |G\B(G)| = rank(G)-1.
2. If H is a non-trivial f.g. subgroup of G then B(H) \subseteq B(G). Hence there is a map H\B(H) -> G\B(G).
It is convenient to define B({1}) to be the empty set, and extend B(-) to commute with infinite ascending chains. Thus B(H) is defined for evey subgroup H of G.

If H and K are f.g. subgroups of G then the natural map
(H cap K)\ B(H cap K) \to H\B(H) \times K\B(K)
is injective and, hence, the HNC holds. Similarly for the SHNC.

The set B(G) is defined as follows.

Let T(G) denote the minimal G-subtree of U. Then G\T(G) is finite.   For each g in G-{1}, T(<g>) is called the axis of g and is homeomorphic to the real line.

Let B(G) denote the set of edges e of T(G) for which there exists some reduced bi-infinite path in T(G) in which e is the <-largest edge.

It is convenient to define T({1}) and B({1}) to be the empty set.

Then B(G) is a G-subset of EU, and for each f.g. subgroup H of G, it is easy to see that T(H) \subseteq T(G) and B(H) \subseteq B(G).

If G has rank at least two, we shall see that B(G) is nonempty. Let g,h be independent elements of G. By replacing g with g-1 if necessary, we may assume that g < 1, and similarly h < 1. Let q be a path in T(G) of shortest possible length (possibly zero) joining T(<g>) to T(<h>), say q joins v in T(<g>) to w in T(<h>). Let p denote the reduced path in T(<g>) joining gv to v, and let r denote the reduced path in T(<h>) joining w to hw. Then   · · ·   g2 p · g p · p · q · r · h r · h2 r · · · is a bi-infinite reduced path in T(G) and it has a <-largest edge, given by the <-largest edge in p · q · r. This shows that B(G) is nonempty if rank(G) > 1.
Consider an arbitrary component T' of T(G)-B(G) and let H be the G-stabilizer of T'.
It remains to show that rank(H) = 1. Notice that H is a free factor of G and that T(H) lies in T'. Hence B(H) lies in T(G)-B(G) and also in B(G). Hence B(H) is empty.
Hence rank(H) < 2.

It remains to show that H \ne {1}. Let v be a vertex in T'. Since G \ne {1}, there exists some infinite reduced ray p in T(G) with initial vertex v. Let N := |G\ET(G)|+1. Let e_1,...,e_s be the G-bridges at distance at most N from v, and sort them so that e_1 > e_2 > ... > e_s.   Consider the least i such that that p crosses the bridge e_i. We can replace the tail of p from e_i onwards with an infinite ray in T(G) formed from edges < e_i and reduce.   After at most s such tail-replacement steps we have a new p that does not cross any of the bridges e_i at distance at most N from v.
Hence the first N edges of p lie in T' and, hence, |ET'| > N-1 = |G\ET(G)|. Hence, there exists g in G and e in ET(G) such that e, ge in ET' and ge \ne e. Then gT' cap T' is nonempty, and, hence, g \in H -{1}. This completes the proof.